

**THE DISPERSIVE APPROACH TO ELECTROWEAK PROCESSES
IN THE BACKGROUND MAGNETIC FIELD**

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We propose a new method to compute amplitudes of electroweak processes in the strong background magnetic field, using $\gamma \rightarrow e^+e^-$ as an example. We show that the *moments* of $\gamma \rightarrow e^+e^-$ width are proportional to the derivatives of photon polarization function at the zero energy. Hence, the pair-production width can be easily calculated from the latter by the inverse Mellin transform. The prospects of our approach are commented.

The electroweak phenomena associated with an intensive background magnetic field are rather rich. Under a background magnetic field, a physical photon can decay into an e^+e^- pair or split into two photons. Such processes are relevant to the attenuation of gamma-rays from pulsars^{1,2}. Similarly, with a sufficient energy, a neutrino can go through the decays $\nu \rightarrow \nu e^+e^-$ and $\nu \rightarrow \nu\gamma$. For processes without charged fermions in the final state, such as $\gamma \rightarrow \gamma\gamma$ or $\nu \rightarrow \nu\gamma$, their decay widths can be expressed as asymptotic series in B for $B < B_c$. However, such asymptotic expansions are not possible for $\nu \rightarrow \nu e^+e^-$ or $\gamma \rightarrow e^+e^-$ since the wave functions of final-state fermions are non-analytic with respect to the magnetic field strength at $B = 0$.

Previously, the photon absorption coefficients due to $\gamma \rightarrow e^+e^-$ were computed in two different ways. One either directly squares the $\gamma \rightarrow e^+e^-$ amplitude using the exact electron (positron) wave function in the background magnetic field⁵, or applies the optical theorem on the photon polarization function $\Pi_{\mu\nu}$ ⁶. In both approaches, the results are valid only for $B < B_c$ and $\omega \sin \theta \gg 2m_e$, where ω is the photon energy and θ is the angle between the magnetic-field direction and the direction of photon propagation. It has been pointed out⁷ that a correct description of $\gamma \rightarrow e^+e^-$ near the pair-production threshold $\omega \sin \theta \approx 2m_e$ is crucial for astrophysical applications. A numerical study taking into account the threshold behavior of $\gamma \rightarrow e^+e^-$ was also carried out⁷. In this work, we re-examine the previous analytic approaches to $\gamma \rightarrow e^+e^-$ ^{5,6}, clarifying their implicit assumptions which lead to incorrect threshold behavior for the above decay.

Let us follow Ref. ⁶ which begins with the proper-time representation^{8,9} of photon polarization function $\Pi_{\mu\nu}$ in the background magnetic field:

$$\begin{aligned} \Pi_{\mu\nu}(q) &= -\frac{e^3 B}{(4\pi)^2} \int_0^\infty ds \int_{-1}^{+1} dv \{ e^{-is\phi_0} [(q^2 g_{\mu\nu} - q_\mu q_\nu) N_0 \\ &\quad - (q_\parallel^2 g_{\parallel\mu\nu} - q_{\parallel\mu} q_{\parallel\nu}) N_\parallel + (q_\perp^2 g_{\perp\mu\nu} - q_{\perp\mu} q_{\perp\nu}) N_\perp] \\ &\quad - e^{-ism_e^2} (1 - v^2) (q^2 g_{\mu\nu} - q_\mu q_\nu) \}, \end{aligned} \quad (1)$$

where

$$\phi_0 = m_e^2 - \frac{1 - v^2}{4} q_\parallel^2 - \frac{\cos(zv) - \cos(z)}{2z \sin(z)} q_\perp^2, \quad (2)$$

with $z = eBs$, and $N_{0,\parallel,\perp}$ trigonometric functions of z and v . Here \parallel and \perp are defined relative to the magnetic-field direction.

The photon dispersion relation is given by $q^2 + \text{Re}\Pi_{\parallel,\perp} = 0$, where $\Pi_{\parallel,\perp} = \epsilon_{\parallel,\perp}^\mu \Pi_{\mu\nu} \epsilon_{\parallel,\perp}^\nu$ with ϵ_{\parallel}^μ and ϵ_{\perp}^μ respectively the photon polarization vectors parallel and perpendicular to the plane spanned by the photon momentum \mathbf{q} and the magnetic field \mathbf{B} . The imaginary part of $\Pi_{\parallel,\perp}$ is related to photon absorption coefficients $\kappa_{\parallel,\perp}$ (i.e. the width of $\gamma \rightarrow e^+e^-$) via $\kappa_{\parallel,\perp} = \text{Im}\Pi_{\parallel,\perp}/\omega$, with ω the photon energy. The authors of Ref. ⁶ analyzed the functions $\Pi_{\parallel,\perp}$ in the limit $\omega \sin \theta \gg 2m_e$ and $B < B_c$. They found $\kappa_{\parallel,\perp} = \frac{1}{2}\alpha \sin \theta (eB/m_e) T_{\parallel,\perp}(\lambda)$, with

$$T_{\parallel,\perp}(\lambda) = \frac{4\sqrt{3}}{\pi\lambda} \int_0^1 dv (1-v^2)^{-1} \left[\left(1 - \frac{1}{3}v^2\right), \left(\frac{1}{2} + \frac{1}{6}v^2\right) \right] K_{2/3} \left(\frac{4}{\lambda} \frac{1}{1-v^2} \right), \quad (3)$$

where $\lambda = \frac{3}{2}(eB/m_e^2)(\omega/m_e) \sin \theta$ and $K_{2/3}$ is the modified Bessel function. Compared to the numerical study⁷, this result is accurate at the higher energy with $\xi \equiv \omega^2 \sin^2 \theta B_c / 2m_e^2 B > 10^3$. However its low energy prediction is problematic. One would expect that $T_{\parallel,\perp}(\lambda)$ vanishes for $\omega \sin \theta$ below the pair production threshold. On the other hand, for $\lambda \ll 1$, $T_{\parallel,\perp} \rightarrow (3/2)^{1/2} \cdot (1/2, 1/4)e^{-4/\lambda}$, which does not behave like a step function. Furthermore $T_{\parallel,\perp}$ is a smooth function of λ , while in actual situation it should contain infinite many sawtooth absorption edges corresponding to higher Landau levels reachable by the increasing photon energies. These discrepancies might have to do with the approximation made in Ref. ⁶ where only the small- s contribution in Eq. (1) is taken into account. However, due to the highly oscillatory behavior of the integrand, it remains unclear how to evaluate the large- s contribution.

Recently, we have developed a technique to deal with the external-field problem, using the analytic properties of physical amplitudes¹⁰. We observe that once the amplitude of a physical process is known in the small momentum (energy) regime, its behavior at arbitrary momentum (energy) is completely determined by the inverse Mellin transform. For the current problem, we have¹⁰

$$\frac{1}{n!} \left(\frac{d^n}{d(\omega^2)^n} \Pi_{\parallel,\perp} \right) \Big|_{\omega^2=0} = \frac{M_{\parallel,\perp}^{1-2n}}{\pi} \int_0^1 dy \cdot y^{n-1} \cdot \left(\kappa_{\parallel,\perp}(y) y^{-1/2} \right), \quad (4)$$

where $y = M_{\parallel,\perp}^2/\omega^2$ with $M_{\parallel,\perp}$ the threshold energies of pair productions^{5,11} given by $M_{\parallel}^2 \sin^2 \theta = 4m_e^2$ and $M_{\perp}^2 \sin^2 \theta = m_e^2 \left(1 + \sqrt{1 + 2B/B_c}\right)^2$. One notes that the imaginary part of $\Pi_{\parallel,\perp}(\omega^2)$ vanishes for the range $0 \leq \omega^2 \leq M_{\parallel,\perp}^2$. This property has been verified in the previous works^{5,11}. Therefore one can effectively set the integration range of Eq. (4) as from $y = 0$ to $y = \infty$. It is then obvious that the derivatives of $\Pi_{\parallel,\perp}$ at the zero energy are proportional to the Mellin transform of $\kappa_{\parallel,\perp} \cdot y^{-1/2} \equiv \kappa_{\parallel,\perp} \cdot \omega / M_{\parallel,\perp}$. Once the l.h.s. of Eq. (4) is calculated, the absorption coefficients $\kappa_{\parallel,\perp}$ can be determined by the inverse Mellin transform.

The l.h.s of Eq. (4) is calculated with a rotation of integration contour $s \rightarrow -is$, which is permissible only for ω below the pair-production threshold. By this rotation, the phase factor $\exp(-is\phi_0)$ in Eq. (1) turns into the more well-behaved factor $\exp(-s\tilde{\phi}_0)$ where $\tilde{\phi}_0$ is obtained from ϕ_0 by the replacement $z \rightarrow -iz$ ($z = eBs$). Furthermore, the trigonometric functions in the integrand $N_{0,\parallel,\perp}$ also turn into the hyperbolic function of z and v . Due to the presence of $\exp(-s\tilde{\phi}_0)$ and

the assumption of a sub-critical magnetic field $B < B_c$, one may obtain a first-approximation for $(d/d\omega^2)^n \Pi_{\parallel,\perp}|_{\omega^2=0}$ by disregarding the large- s contribution in the integral of Eq. (1) and its derivatives. This amounts to approximating, for example, $\cosh(z)$ there by the power series $1 + z^2/2 + \dots$. With this approximation, we arrive at

$$\begin{aligned} \frac{1}{n!} \left(\frac{d^n}{d(\omega^2)^n} \Pi_{\parallel,\perp} \right) \Big|_{\omega^2=0} &= \frac{2\alpha m_e^2}{\pi} \left(\frac{B^2 \sin^2 \theta}{3B_c^2 m_e^2} \right)^n \frac{\Gamma(3n-1)\Gamma^2(2n)}{\Gamma(n)\Gamma(4n)} \\ &\times \left(\frac{6n+1, 3n+1}{4n+1} \right) + \dots, \end{aligned} \quad (5)$$

where the neglected terms are suppressed by the factor $(B/B_c)^2$. Combining Eqs. (4) and (5), one obtains the photon absorption coefficients $\kappa_{\parallel,\perp}$ by the inverse Mellin transform¹⁰:

$$\begin{aligned} \kappa_{\parallel} &= \frac{\alpha m_e^2}{i\pi\omega} \int_{-i\infty+a}^{+i\infty+a} ds(\lambda')^{2s} \frac{\Gamma(3s)\Gamma^2(2s)}{\Gamma(s)\Gamma(4s)} \frac{1}{3s-1} \times \frac{6s+1}{4s+1}, \\ \kappa_{\perp} &= \frac{2\alpha m_e^2}{i\pi\omega} \frac{1}{1 + \sqrt{1 + 2B/B_c}} \int_{-i\infty+a}^{+i\infty+a} ds(\lambda'')^{2s} \frac{\Gamma(3s)\Gamma^2(2s)}{\Gamma(s)\Gamma(4s)} \frac{1}{3s-1} \times \frac{3s+1}{4s+1}, \end{aligned} \quad (6)$$

where a is any real number greater than $1/3$; while $\lambda' = (\omega \sin \theta B / \sqrt{3} m_e B_c)$ and $\lambda'' = \lambda' \cdot (1 + \sqrt{1 + 2B/B_c})/2$. It can be shown rigorously that¹⁰ $\kappa_{\parallel,\perp}$ computed in this way are equivalent to results of Tsai and Erber given in Eq. (3), except on some trivial kinematic factors.

The work on improving Eq. (5) and consequently the threshold behavior of $\kappa_{\parallel,\perp}$ is in progress¹². We have found that the large- s contribution which is disregarded in the first-approximation becomes important in the higher derivatives of $\Pi_{\parallel,\perp}$. Taking into account this contribution is crucial to obtain correct threshold behaviors for $\kappa_{\parallel,\perp}$.

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